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Flow Equations for Uplifting Half-Flat to Spin(7) Manifolds

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Abstract

In this supplement to [1], we discuss the uplift of half-flat six-folds to Spin(7) eight-folds by fibration of the former over a product of two intervals. We show that the same can be done in two ways - one, such that the required Spin(7) eight-fold is a double G_2 seven-fold fibration over an interval, the G_2 seven-fold itself being the half-flat six-fold fibered over the other interval, and second, by simply considering the fibration of the half-flat six-fold over a product of two intervals. The flow equations one gets are an obvious generalization of the Hitchin's flow equations (to obtain seven-folds of G_2 holonomy from half-flat six-folds [2]). We explicitly show the uplift of the Iwasawa using both methods, thereby proposing the form of new Spin(7) metrics. We give a plausibility argument ruling out the uplift of the Iwasawa manifold to a Spin(7) eight fold at the "edge", using the second method. For $Spin(7)$ eight-folds of the type $X_7 \times S^1$, X_7 being a seven-fold of $SU(3)$ structure, we motivate the possibility of including elliptic functions into the "shape deformation" functions of seven-folds of $SU(3)$ structure of [1] via some connections between elliptic functions, the Heisenberg group, theta functions, the already known $D7$-brane metric[3] and hyper-Kähler metrics

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obtained in twistor spaces by deformations of Atiyah-Hitchin manifolds by a Legendre transform in [4].

It is known that manifolds with G_2 and Spin(7) holonomies, are very useful in getting minimal amount of supersymmetry after compactification of seven and eight dimensions, respectively, in string/M-theory [9]. In the past few years, half-flat manifolds have been shown to be relevant to flux compactifications in string theory (see [5, 6] and references therein). In [1], using the results of [7], we explicitly showed how to uplift the Iwasawa manifold, an example of a half-flat manifold, to seven-folds of either G_2-holonomy or $SU(3)$ structure. In this short note, we show how to uplift half-flat manifolds to Spin(7) eight-folds. We will be following the notations of [1]. We also give plausibility arguments in favor of inclusion of the Weierstrass elliptic function in $Spin(7)$-metrics of the type $X_7 \times S^1$, X_7 being seven-folds of $SU(3)$ structure.

Spin(7) folds are characterized by a self-dual closed (and hence co-closed) Cayley four-form (with the additional constraint that the $\hat{A}$-genus of the eight-fold equals unity [8]). We will begin with the construction of a Spin(7) eight-fold as a double G_2-fibration over an interval. We then go on to constructing a Spin(7) eight-fold as a fibration of a half-flat over the product of two intervals.

Spin(7) as a Double G_2-fibration over an interval

Let us begin with the following construction of a Spin(7) eight-fold $X_8^{\mathrm{Spin}(7)}$:

$$
\begin{array}{cccc}
I_{t_1} & & & \\
\uparrow & & & \\
| & X_7^{G_2}(t_2) & \longrightarrow I_{t_2} & \\
| & & M_6 & \\
| & & & \\
X_8^{\mathrm{Spin}(7)} & \longrightarrow & I_{t_2} & \\
& X_7^{G_2}(t_1) & & \\
& | & & \\
& \downarrow M_6 & & \\
& I_{t_1} & & \\
& & Fig.1 &
\end{array}
$$

In other words, one has a "double" G_2-fibration structure in the sense that the Spin(7) eight-fold is a fibration of a $G_2(t_1)$-manifold over an interval $I(t_2)$ *as well as* a $G_2(t_2)$-manifold over an interval $I(t_1)$, where the $G_2(t_i)$-manifold is itself a fibration of a half-flat six-fold over an interval $I(t_i)$ (via the Hitchin's construction [2]). Given a half-flat manifold $M_6(\Omega, J)$, the Hitchin's construction involves (w.r.t. seven dimensions) a closed and co-closed three-form $\phi_3 = J \wedge dt + \Omega_+(t)$, $\Omega_+ \equiv \mathrm{Re}(\Omega)$, one can write down the following Cayley four-form ϕ_4:

$$
\begin{aligned}
&\phi_4 = \alpha_1 *_7 \phi_3^1 + \alpha_2 *_7 \phi_3^2 + \tilde{\alpha}_1 \phi^1 \wedge dt_2 + \tilde{\alpha}_2 \phi_3^2 \wedge dt_1 \\
&+\beta_1 \Omega_+ \wedge dt_1 + \beta_2 \Omega_+ \wedge dt_2 + \gamma_1 \Omega_- \wedge dt_1 + \gamma_2 \Omega_- \wedge dt_2 + \delta J \wedge J + \epsilon J \wedge dt_1 \wedge dt_2.
\end{aligned}
\tag{1}
$$

The self-duality condition would thus imply:

$$
\begin{aligned}
&\alpha_1\Big(J\wedge dt_1\wedge dt_2+\Omega_+\wedge dt_2\Big)+\alpha_2\Big(J\wedge dt_1\wedge dt_2-\Omega_+\wedge dt_1\Big)\\
&+\tilde{\alpha}_1\Big(\frac{1}{2}J\wedge J+\Omega_-\wedge dt_1\Big)+\tilde{\alpha}_2\Big(-\frac{1}{2}J\wedge J-\Omega_-\wedge dt_2\Big)\\
&-\beta_1\Omega_-\wedge dt_2+\beta_2\Omega_-\wedge dt_1+\gamma_1\Omega_+\wedge dt_2-\gamma_2\Omega_+\wedge dt_1\\
&+2\delta J\wedge dt_1\wedge dt_2+\frac{\epsilon}{2}J\wedge J\\
&=\alpha_1\Big(\frac{1}{2}J\wedge J+\Omega_-\wedge dt_1\Big)+\alpha_2\Big(\frac{1}{2}J\wedge J+\Omega_-\wedge dt_2\Big)\\
&+\tilde{\alpha}_1\Big(J\wedge dt_1\wedge dt_2+\Omega_+\wedge dt_2)+\tilde{\alpha}_2(-J\wedge dt_1\wedge dt_2+\Omega_+\wedge dt_1\Big)\\
&+\beta_1\Omega_+\wedge dt_1+\beta_2\Omega_+\wedge dt_2+\gamma_1\Omega_-\wedge dt_1+\gamma_2\Omega_-\wedge dt_2\\
&\delta J\wedge J+\epsilon J\wedge dt_1\wedge dt_2,
\end{aligned}
\tag{2}
$$

implying:

$$
\begin{aligned}
&\alpha_1+\alpha_2+2\delta=\tilde{\alpha}_1-\tilde{\alpha}_2+\epsilon,\\
&\alpha_1+\gamma_1=\tilde{\alpha}_1+\beta_2,\ -\alpha_2-\gamma_2=\tilde{\alpha}_2+\beta_1.
\end{aligned}
\tag{3}
$$

Hence, the following is the form of the self-dual four-form:

$$
\begin{aligned}
&\phi_4=\alpha_1 *_7\phi^1_3+\alpha_2 *_7\phi^2_3+\tilde{\alpha}_1\phi^1\wedge dt_2+\tilde{\alpha_2}\phi^2_3\wedge dt_1\\
&+\beta_1\Omega_+\wedge dt_1+\beta_2\Omega_+\wedge dt_2+(\tilde{\alpha}_1+\beta_2-\alpha_1)\Omega_-\wedge dt_1\\
&-(\alpha 2+\tilde{\alpha}_2+\beta_1)\Omega_-\wedge dt_2+\delta J\wedge J+(\alpha_1+\alpha_2+2\delta+\tilde{\alpha}_2-\tilde{\alpha}_1)J\wedge dt_1\wedge dt_2.
\end{aligned}
\tag{4}
$$

For (4) to be a Cayley four-form, it must satisfy the condition that it is closed, which on using:

$$
\begin{aligned}
&d *_7\phi^1_3=(\hat{d}+dt_1\wedge\frac{\partial}{\partial t_1}+dt_2\wedge\frac{\partial}{\partial t_2}) *_7\phi^1_3=dt_2\wedge\frac{\partial *_7\phi^1_3}{\partial t_2}\\
&d *_7\phi^2_3=(\hat{d}+dt_1\wedge\frac{\partial}{\partial t_1}+dt_2\wedge\frac{\partial}{\partial t_2}) *_7\phi^2_3=dt_1\wedge\frac{\partial *_7\phi^2_3}{\partial t_2}\\
&d\phi^1_3=dt_2\wedge\frac{\partial\phi^1_3}{\partial t_2},\ d\phi^2_3=dt_2\wedge\frac{\partial\phi^2_3}{\partial t_1},
\end{aligned}
\tag{5}
$$

implies:

$$
\begin{aligned}
&d\phi_4|_{\hat{d}J\wedge J=\hat{d}\Omega_+=0} = \alpha_1\left(dt_2\wedge\frac{\partial J}{\partial t_2}\wedge J + dt_2\wedge\frac{\partial\Omega_-}{\partial t_2}\wedge dt_1\right)\\
&+\alpha_2\left(dt_1\wedge\frac{\partial J}{\partial t_1}\wedge J + dt_1\wedge\frac{\partial\Omega_-}{\partial t_1}\wedge dt_1\right) + \beta_1(dt_2\wedge\frac{\partial\Omega_+}{\partial t_2}\wedge dt_1) + \beta_2(dt_1\wedge\frac{\partial\Omega_+}{\partial t_1}\wedge dt_2)\\
&+(\tilde{\alpha}_1+\beta_2-\alpha_1)\left(\hat{d}\Omega_1\wedge dt_1 + dt_2\wedge\frac{\partial\Omega_-}{\partial t_2}\wedge dt_1\right)\\
&-(\alpha_2+\tilde{\alpha}_2+\beta_1)\left(\hat{d}\Omega_1\wedge dt_2 + dt_1\wedge\frac{\partial\Omega_-}{\partial t_1}\wedge dt_2\right) + 2\delta\left(dt_1\wedge\frac{\partial J}{\partial t_1}\wedge J + dt_2\wedge\frac{\partial J}{\partial t_2}\wedge J\right)\\
&+(\alpha_1+\alpha_2+2\delta+\tilde{\alpha}_2-\tilde{\alpha}_1)\hat{d}J\wedge dt_1\wedge dt_2 = 0. \qquad (6)
\end{aligned}
$$

One thus gets the following flow equations:

$$
\begin{aligned}
&(\alpha_1+2\delta)\frac{\partial J}{\partial t_2}\wedge J = (\alpha_2+\tilde{\alpha}_2+\beta_1)\hat{d}\Omega_-,\\
&(2\delta+\alpha_2)\frac{\partial J}{\partial t_1}\wedge J = (\alpha_1-\beta_2+\tilde{\alpha}_1)\hat{d}\Omega_-,\\
&(\tilde{\alpha}_1+\tilde{\beta}_2)\frac{\partial\Omega_-}{\partial t_2} + (\tilde{\alpha}_2+\beta_1)\frac{\partial\Omega_-}{\partial t_1} + \beta_1\frac{\partial\Omega_+}{\partial t_2} - \beta_2\frac{\partial\Omega_+}{\partial t_1} + (\alpha_1+\alpha_2+2\delta+\tilde{\alpha}_2-\tilde{\alpha}_1)\hat{d}J = 0.
\end{aligned}
\qquad (7)
$$

Let us use the flow equations of (7) to explicitly uplift the Iwasawa manifold to a Spin(7) eight-fold, working with the standard complex structure limit of the Iwasawa, and consider its deformation of the type:

$$
J(t_1,t_2) = e^{a(t_1,t_2)}e^{12} + e^{b(t_1,t_2)}e^{34} + e^{c(t_1,t_2)}e^{56}, \qquad (8)
$$

and

$$
\Omega(t_1,t_2) = e^{\frac{(a+b+c)(t_1,t_2)}{2}}(e^1+ie^2)\wedge(e^3+ie^4)\wedge(e^5+ie^6). \qquad (9)
$$

Then the flow equations (7) imply:

-
$$
\frac{\partial a}{\partial t_1} = \frac{\partial b}{\partial t_1} = -\frac{\partial c}{\partial t_1},\ 2\frac{\partial a}{\partial t_1}e^{a+b} = 4\xi_1 e^{\frac{a+b+c}{2}}, \qquad (10)
$$

where $\xi_1 \equiv \frac{\alpha_1-\beta_2+\tilde{\alpha}_1}{2\delta+\alpha_2}$, which could be satisfied by equality of $a, b, -c$, and:

$$a(t_1,t_2) = \frac{2}{3} ln\Big(3\xi_1 e^{\lambda_1} t_1 + f_2(t_2)\Big), \tag{11}$$

λ_1 being a linear combination of the integration constants that would appear in the integration of the first set of equations in (10);

- similarly,

$$a(t_1,t_2) = \frac{2}{3} ln\Big(3\xi_2 e^{\lambda_1} t_2 + f_1(t_1)\Big), \tag{12}$$

where $\xi_2 \equiv \frac{\alpha_2+\tilde{\alpha_2}+\beta_1}{\alpha_1+2\delta}$.

-

$$\begin{aligned}
&\frac{(\tilde{\alpha}_1+\beta_2)}{2}\frac{\partial a}{\partial t_2} + \frac{(\tilde{\alpha}_2+\beta_1)}{2}\frac{\partial a}{\partial t_1} = 0,\\
&\Leftrightarrow (\tilde{\alpha}_1+\beta)\xi_2 + (\tilde{\alpha}_2+\beta_1)\xi_1 = 0,\\
&e^{\frac{a+b+c}{2}}\left(\frac{\beta_1}{2}\frac{\partial a}{\partial t_2} - \frac{\beta_2}{2}\frac{\partial a}{\partial t_1}\right) = (\alpha_1+\alpha_2+2\delta+\tilde{\alpha}_2-\tilde{\alpha}_1)e^c,\\
&\Leftrightarrow (\beta_1\xi_2-\beta_2\xi_1)e^{-\frac{3a}{2}+\lambda_1} = (\alpha_1+\alpha_2+2\delta+\tilde{\alpha}_2-\tilde{\alpha}_1)e^{-a+\lambda_3},\\
&\Rightarrow \beta_1\xi_2-\beta_2\xi_1 = \alpha_1+\alpha_2+2\delta+\tilde{\alpha}_2-\tilde{\alpha}_1.
\end{aligned} \tag{13}$$

Hence, the metric corresponding to the Spin(7) eight-fold obtained by uplifting the Iwasawa manifold via the flow equations of (7) such that the eight-fold is a double G_2-fibration over an interval, is given by the following solutions to (10) to (13):

$$ds_8^2 = ds_{I\times I}^2(t_1,t_2) + (1+\xi_1 t_1+\xi_2 t_2)^{\frac{2}{3}}(|dz|^2+|dv|^2) + \frac{1}{(1+\xi_2 t_1+\xi_2 t_2)^{\frac{2}{3}}}|du-zdv|^2, \tag{14}$$

with the constraints:

$$\begin{aligned}
&\beta_1\xi_2-\beta_2\xi_1 = \alpha_1+\alpha_2+2\delta+\tilde{\alpha}_2-\tilde{\alpha}_1,\\
&\frac{\xi_1}{\xi_2} = -\frac{\tilde{\alpha}_1+\beta_2}{\tilde{\alpha}_2+\beta_1}, \text{ or } \tilde{\alpha}_1 = -\beta_2,\ \tilde{\alpha}_2 = -\beta_1.
\end{aligned} \tag{15}$$

Notice that (14) has the required double G_2-fibration structure of Fig.1 by noting that (14) is made up of:

$$ds_7^2(t_1, \text{given } t_2) = dt_1^2 + (\xi_1' + \xi_1 t_1)^{\frac{2}{3}}(|dz|^2 + |dv|^2) + \frac{1}{(\xi_1' + \xi_1 t_1)^{\frac{2}{3}}}|du - zdv|^2, \tag{16}$$

and

$$ds_7^2(t_2, \text{given } t_1) = dt_2^2 + (\xi_2' + \xi_2 t_2)^{\frac{2}{3}}(|dz|^2 + |dv|^2) + \frac{1}{(\xi_2' + \xi_2 t_2)^{\frac{2}{3}}}|du - zdv|^2, \tag{17}$$

which are the two-parameter G_2-metrics of [1] [2] The metric of (14) also thus has G_2-holonomy[3].

Spin(7) as a fibration of a Half-flat over $I \times I$

Let us now consider the following fibration structure:

$$\begin{array}{ccc} X_8^{\text{Spin}(7)} & \longrightarrow & I_{t_1} \times I_{t_2} \\ & M_6 & \end{array} \tag{18}$$

Let us assume that the Cayley four-form is given by:

$$\phi_4 = a_1 \Omega_- \wedge dt_1 + a_2 \Omega_- \wedge dt_2 + b_1 \Omega_+ \wedge dt_1 + b_2 \Omega_+ \wedge dt_2 + cJ \wedge J + fJ \wedge dt_1 \wedge dt_2. \tag{19}$$

One hence gets:

$$*_8\phi_4 = a_1 \Omega_+ \wedge dt_2 - a_2 \Omega_+ \wedge dt_1 - b_1 \Omega_- \wedge dt_2 + b_2 \Omega_- \wedge dt_1 + 2cJ \wedge dt_1 \wedge dt_2 + \frac{f}{2} J \wedge J, \tag{20}$$

implying that for $\phi_4 = *_8\phi_4$,

$$a_1 = b_1, \; a_2 = -b_1, \; c = \frac{d}{2}. \tag{21}$$

The required Cayley four-form is:

$$\phi_4 = a_1 \Omega_- \wedge dt_1 + a_2 \Omega_- \wedge dt_2 - a_2 \Omega_+ \wedge dt_1 + a_1 \Omega_+ \wedge dt_2 + \frac{f}{2} J \wedge J + fJ \wedge dt_1 \wedge dt_2. \tag{22}$$

[2] In [1], however, one had set $\xi_i' = 1$.

[3] We thank J.Maldacena for pointing this out.

Finally, the condition that ϕ_4 of (22) is closed gives:

$$
\begin{aligned}
d\phi_4 &= a_1\Big(\hat{d}\Omega_- \wedge dt_1 + dt_2 \wedge \frac{\partial\Omega_-}{\partial t_2} \wedge dt_1\Big) + a_2\Big(\hat{d}\Omega_- \wedge dt_2 + dt_1 \wedge \frac{\partial\Omega_-}{\partial t_1} \wedge dt_2\Big)\\
&+a_1\Big(\hat{d}\Omega_+ \wedge dt_2 + dt_1 \wedge \frac{\partial\Omega_+}{\partial t_1} \wedge dt_2\Big) - a_2\Big(\hat{d}\Omega_+ \wedge dt_1 + dt_2 \wedge \frac{\partial\Omega_-}{\partial t_2} \wedge dt_1\Big)\\
&+f\Big(\hat{d}J \wedge J + dt_1 \wedge \frac{\partial J}{\partial t_1} \wedge J + dt_2 \wedge \frac{\partial J}{\partial t_2} \wedge J\Big) = 0.
\end{aligned}
\tag{23}
$$

Using that for half-flat manifolds, $\hat{d}J \wedge J = \hat{d}\Omega_+ = 0$, one thus gets the following flow equations:

$$
\begin{aligned}
&a_1\ \hat{d}\Omega_- = -f\ \frac{\partial J}{\partial t_1} \wedge J,\\
&a_2\ \hat{d}\Omega_- = -f\ \frac{\partial J}{\partial t_2} \wedge J,\\
&-a_1\ \frac{\partial\Omega_+}{\partial t_2} + a_2\ \frac{\partial\Omega_-}{\partial t_1} + a_2\ \frac{\partial\Omega_+}{\partial t_2} + a_1\ \frac{\partial\Omega_+}{\partial t_2} = f\ \hat{d}J.
\end{aligned}
\tag{24}
$$

One can again show that one can explicitly uplift the Iwasawa manifold to a Spin(7) eight-fold at standard complex structure limit of the Iwasawa and consider its deformation of the type as given in (8) and (9). The set of equations that one gets from (24), are:

$$
\begin{aligned}
&\frac{\partial a}{\partial t_i} = \frac{\partial b}{\partial t_i} = -\frac{\partial c}{\partial t_i},\\
&\frac{\partial a}{\partial t_1} = -\frac{2a_1}{f}e^{-\frac{3a}{2}+\lambda_1},\ \frac{\partial a}{\partial t_2} = -\frac{2a_2}{f}e^{-\frac{3a}{2}+\lambda_1},\\
&\Big(\frac{a_1}{2}\frac{\partial a}{\partial t_2} - a_2\frac{\partial a}{\partial t_1}\Big)e^{\frac{a+b+c}{2}} = 0,\ \Big(-\frac{a_2}{2}\frac{\partial a}{\partial t_2} - \frac{a_1}{2}\frac{\partial a}{\partial t_1}\Big)e^{\frac{a+b+c}{2}} = fe^c,
\end{aligned}
\tag{25}
$$

which are satisfied by:

$$
\begin{aligned}
&a(t_1,t_2) = \frac{2}{3}ln\Big(1 + 3\frac{a_1e^{\lambda_1}}{f}t_1 + 3\frac{a_2e^{\lambda_2}}{f}t_2\Big),\\
&\text{with } a_1^2 + a_2^2 = f^2.
\end{aligned}
\tag{26}
$$

Thus, the metric for the Spin(7) eight-fold is:

$$
\begin{aligned}
ds_8^2 &= ds^2_{I\times I}(t_1,t_2) + \Big(1 + \frac{a_1}{\sqrt{a_1^2+a_2^2}}t_1 + \frac{a_2}{\sqrt{a_1^2+a_2^2}}t_2\Big)^{\frac{2}{3}}(|dz|^2 + |dv|^2)\\
&+\Big(1 + \frac{a_1}{\sqrt{a_1^2+a_2^2}}t_1 + \frac{a_2}{\sqrt{a_1^2+a_2^2}}t_2\Big)^{-\frac{2}{3}}|du - vdz|^2.
\end{aligned}
\tag{27}
$$

Again, the metric also has G_2-holonomy.

However, it is unlikely to be able to uplift the Iwasawa to a Spin(7) eight-fold at the "edge". One notes that at the edge (See [5] and references therein), the one-forms, incorporating $t_{1,2}$-dependent deformations, are: $\alpha = -e^{\mathcal{A}(t_1,t_2)}f^1, \beta = e^{\mathcal{B}(t_1,t_2)}(f^3 + if^4), \gamma = e^{\mathcal{C}(t_1,t_2)}(e^5 + ie^6)$, and $J = \frac{i}{2}(\alpha \wedge \bar{\alpha} + \beta \wedge \bar{\beta} + \gamma \wedge \bar{\gamma}$ and $\Omega = \alpha \wedge \beta \wedge \gamma)$. The one-forms $f^i, i = 1, ..., 4$ are defined via $f^i = P^i_j e^j$, where $P \in SO(4)$ matrix, and one write it as $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$, where $X, Y \in SU(2)$, i.e., $\begin{pmatrix} P^1_1 & P^1_2 & 0 & 0 \\ P^2_1 & P^2_2 & 0 & 0 \\ 0 & 0 & P^3_3 & P^3_4 \\ 0 & 0 & P^4_3 & P^4_4 \end{pmatrix}$, where $P^1_1 P^2_2 - P^1_2 P^2_1 = P^3_3 P^4_4 - P^3_4 P^4_3 = 1$. The flow equations $\hat{d}\Omega_- = -\frac{\partial J}{\partial t_1} \wedge J = -\frac{\partial J}{\partial t_2} \wedge J$ implies $\mathcal{A}$ gives the same result after differentiation w.r.t. t_1 or t_2. Unlike the standard complex structure limit, there are common components to Ω_+ and Ω_- in the edge. One can show that the other flow equation $a_1 \frac{\partial \Omega_-}{\partial t_2} - a_2 \frac{\partial \Omega_-}{\partial t_1} - a_2 \frac{\partial \Omega_+}{\partial t_2} - a_1 \frac{\partial \Omega_+}{\partial t_1} + f\hat{d}J = 0$ becomes:

$$
\begin{aligned}
& e^{\mathcal{A}+\mathcal{B}+\mathcal{C}} \Bigg[\Big(\mathcal{D}_-(P^1_1 P^3_3 + P^2_1 P^4_3) - \mathcal{D}_+(P^2_1 P^3_3 - P^1_1 P^4_3) \Big) e^{136} \\
& + \Big(\mathcal{D}_-(P^1_1 P^3_4 + P^2_1 P^4_4) - \mathcal{D}_+(P^2_1 P^3_4 - P^1_1 P^4_4) \Big) e^{146} \\
& + \Big(\mathcal{D}_-(P^1_2 P^3_3 + P^2_2 P^4_3) - \mathcal{D}_+(P^2_2 P^3_3 - P^1_2 P^4_3) \Big) e^{236} \\
& + \Big(\mathcal{D}_-(P^1_2 P^3_4 + P^2_2 P^4_4 - \mathcal{D}_+(P^2_2 P^3_4 - P^1_2 P^4_4) \Big) e^{246} \\
& + \Big(\mathcal{D}_-(-P^1_1 P^4_3 + P^2_1 P^3_3) - \mathcal{D}_+(-P^1_1 P^3_3 - P^2_1 P^4_3) \Big) e^{135} \\
& + \Big(\mathcal{D}_-(-P^1_1 P^4_4 + P^2_1 P^3_4) - \mathcal{D}_+(-P^1_1 P^3_4 - P^2_1 P^4_4) \Big) e^{145} \\
& + \Big(\mathcal{D}_-(-P^1_2 P^4_3 + P62_2 P^3_3) - \mathcal{D}_+(-P^1_2 P^3_3 - P^2_2 P^4_3) \Big) e^{235} \\
& + \Big(\mathcal{D}_-(-P^1_2 P^4_4 + P^2_2 P^3_4) - \mathcal{D}_+(-P^1_2 P^3_4 - P^2_2 P^4_4) \Big) e^{246} \Bigg] \\
& = -e^{2\mathcal{C}} \Big(e^{135} + e^{425} - e^{614} - e^{623} \Big),
\end{aligned}
\tag{28}
$$

(where $\mathcal{D}_+ \equiv \left(a_2\frac{\partial}{\partial t_2} + a_1\frac{\partial}{\partial t_1}\right)\mathcal{ABC}$ and $\mathcal{D}_- \equiv \left(a_1\frac{\partial}{\partial t_2} - a_2\frac{\partial}{\partial t_1}\right)\mathcal{ABC}$), which implies that one will overconstrain the matrix P (from the first set of flow equations, one sees that $\mathcal{D}_- = 0$, and hence one gets from (28), eight equations in the six parameters P^i_j). Hence, the uplift of the edge to a Spin(7) is quite likely to be impossible.

Possibility of introducing Weierstrass elliptic Functions in $Spin(7)$ eight-folds including an S^1

We now give some very compelling evidence in support of the possibility of inclusion of Weierstrass elliptic functions in those $Spin(7)$ eight-folds which are of the type $X_7 \times S^1$, X_7 being a seven fold of $SU(3)$ structure.

Seven-dimensional manifolds of G_2 holonomy or $SU(3)$ structure are required for getting $\mathcal{N} = 1$ supersymmetry in four dimensions from the eleven dimensional M-theory. Similarly, $Spin(7)$ eight-folds are required for getting $\mathcal{N} = 1$ supersymmetry in three dimensions from (the eleven dimensional) M theory. One could explore the option of getting compact $Spin(7)$ uplifts using S^1's instead of intervals by using the same flow equations as derived in this paper, but further demanding periodicity w.r.t. t_1 and t_2, of the solutions.

Assuming the existence of $Spin(7)$ eight-folds of the type $X_7 \times S^1$, one could first argue the existence of a G_2-structure by noting the existence of a singlet in the decomposition under $G_2 \subset Spin(7)$ of the $\mathbf{8}$ in the fundamental spinorial representation ($\mathbf{8} \rightarrow \mathbf{7}+\mathbf{1}$). Further, assuming that Majorana-Weyl spinors ($\xi = \xi^+ \oplus \xi^-$, the $\pm$ signs referring to the chiralities) on the $Spin(7)$ eight-fold are nowhere vanishing, there is a further reduction of the structure group to $G_{2+} \cap G_{2-} = SU(3)$, the two G_2's corresponding to the two chiralities of $\xi^{\pm}$.

Having established the connection between (the use of) $Spin(7)$ eight-folds and seven-folds with $SU(3)$-structure, let us now move to the main theme of this section - the possibility of inclusion of

Weierstrass elliptic functions in seven-folds with $SU(3)$ structure and thereby $Spin(7)$ eight-folds of the type $X_7 \times S^1$.

Using the results of [7], explicit metrics for seven-folds with $SU(3)$ structure were obtained. The "shape" deformation functions "$A(z, \bar{z}; v, \bar{v})$" and "$B(z, \bar{z}; v, \bar{v})$", as indicated in [1], could also be related to elliptic functions - the seven dimensional $SU(3)$ structure does not impose too many constraints if one allows wrapped $M5$-branes in the analysis. The following are some interesting connections between some concepts, thereby motivating further the idea of having singular uplifts to seven dimensions of the Iwasawa manifold, involving elliptic functions:

- The $D7$-brane metric (relevant to "cosmic strings" in [3]) is given by:

$$ds_{10}^2 = ds_8^2 + \tau_2(z)|\eta(\tau(z))|^4|z|^{-\frac{N}{6}}|dz|^2, \tag{29}$$

  where $\tau_2 \equiv$Imτ, $\tau = a + ie^{-\phi}$, $a \equiv$ axion and $\phi \equiv$dilaton, $\eta \equiv$ Dedekind eta function (see below), $N \equiv$ the number of $D7$-branes, and z is the complex coordinate transverse to the $D7$-brane. Hence, one has one explicit example of a metric involving η, which is related to theta functions, as indicated below.

- The Jacobi theta function [10] function defined for two complex variables z and τ where Im$\tau > 0$: $\vartheta(z : \tau) = \sum_{-\infty}^{\infty} e^{i\pi n^2\tau + 2\pi i n z}$. The theta function is related to the Dedekind eta function via: $\vartheta(0; \tau) = \frac{\eta^2(\frac{\tau+1}{2})}{\eta(\tau+1)}$.

- The Weierstrass elliptic function [10] is a doubly periodic function with periods $2\omega_1$ and $2\omega_2$ such that $\omega_1\omega_2$ is not real:

$$\mathcal{P}(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{m,n\in\mathbf{Z}\backslash(0,0)} \left(\frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2}\right).$$

  $\mathcal{P}$ satisfies the following cubic equation:

$$\left(\frac{d\mathcal{P}(z;\tau)}{dz}\right)^2 = \mathcal{P}^3(z;\tau) - g_2\mathcal{P} - g_3$$

where $g_2 = 60\sum_{m,n\in\mathbf{Z}\backslash(0,0)}\frac{1}{(2m\omega_1+2n\omega_2)^4},\ g_3 = 140\sum_{m,n\in\mathbf{Z}\backslash(0,0)}\frac{1}{(2m\omega_1+2n\omega_2)^6}$. This is an equation of a torus, which could be related to the Riemann surface that is referred to in [7]. The torus degenerates, i.e., becomes singular along the discriminant locus given by: $\Delta = g_2^3 - 27g_3^2 = 0$. There is the following relation between the discriminant locus and the Dedekind eta function: $\Delta = (2\pi)^{12}\eta^{24}$. The following relations are true: $\mathcal{P}(z;\omega_1 = 1, \omega_2 = \tau) = -\frac{d^2}{dz^2}\vartheta_{11}(z;\tau)+$ constant $= \pi^2\vartheta^2(0;\tau)\vartheta_{10}^2(0;\tau)\frac{\vartheta_{10}^2(z;\tau)}{\vartheta_{11}^2(z;\tau)}+e_2(\tau)$, where e_2 is one of the three roots $e_{i=1,2,3}$, of the cubic equation $4t^3 - g_2t - g_3 = 0$, and $\mathcal{P}(\omega_1) = e_1,\ \mathcal{P}(\omega_2) = e_2,\ \mathcal{P}(-\omega_1-\omega_2) = e_3$[4]

- Consider a holomorphic function $f(z)$ and $a, b \in \mathbf{R}$. Define two operators S_a and T_b as follows:$(S_a f)(z) = f(z+a),\ (T_b f)(z) = e^{i\pi b^2\tau+2i\pi bz}f(z+b\tau)$. Then S, T and a phase factor form the generators of the nilpotent Heisenberg group central to the group-theoretic way of understanding the Iwasawa manifold. If $U(\lambda\in\mathbf{C}, a, b) \in H \equiv$Heisenberg group, then

$$U(\lambda, a, b)f(z) = \lambda(S_a\circ T_b f)(z) = \lambda e^{i\pi b^2\tau+2i\pi bz}f(z+b\tau+a),$$

  and U is referred to as the theta representation of the Heisenberg group [11].

- (Inverse) Elliptic functions, as shown in [4], naturally figure in the hyper-Kähler metrics in twistor spaces obtained by deformations of Atiyah-Hitchin spaces and Legendre transform. Lets elaborate upon this a little.

  Deformations of Atiyah-Hitchin manifolds (written as hypersurface in $\mathbf{C}^3$: $x^2 + y^2 z = 1$) of the type $x^2 z + (yz+a)^2 = z^2 + a^2$ have been considered. The twistor three-folds are obtained as holomorphic sections $\Gamma(\mathcal{O}_{\mathbf{CP}^1}(4))$ with the following similar equation: $x^2(\zeta)\eta(\zeta) + (y(\zeta)\eta(\zeta) + p(\zeta))^2 = \eta(\zeta) + p^2(\zeta)$, where η is $\Gamma(\mathcal{O}_{\mathbf{CP}^1}(4))$ and the deformation p is $\Gamma(\mathcal{O}_{\mathbf{CP}^1}(2))$, and ζ is a $\mathbf{CP}^1$-valued coordinate. The 'reality involution': $\bar{\eta}^{(2m)}(\zeta) = (-)^m(\bar{\zeta})^{(2m)}(-\frac{1}{\zeta})$, where $\eta^{(2m)}(\zeta)$ is

[4]The e_i's are given by: $e_1(\tau) = \frac{\pi^2}{3}(\vartheta^4(0;\tau)+\vartheta_{01}^4(0;\tau))$, $e_2(\tau) = -\frac{\pi^2}{3}(\vartheta^4(0;\tau)+\vartheta^4(0;\tau))$ $e_3(\tau) = \frac{\pi^2}{3}(\vartheta_{10}^4(0;\tau)-\vartheta^4(0;\tau))$, where the three other theta functions are defined as: $\vartheta_{01}(z;\tau) = \vartheta(z+\frac{1}{2};\tau)$ $\vartheta_{10}(z;\tau) = e^{i\frac{\pi}{4}+i\pi z}\vartheta(z+\frac{\tau}{2};\tau)$ $\vartheta_{11}(z;\tau) = e^{i\frac{\pi}{4}+i\pi(z+\frac{1}{2})}\vartheta(z+\frac{\tau+1}{2};\tau)$.

$\Gamma(\mathcal{O}_{\mathbf{CP^1}}(2m))$, implies that $\eta(\zeta)$ has five independent parameters, i.e., $\eta(\zeta) = z + v\zeta + w\zeta^2 - \bar{v}\zeta^3 + \bar{z}\zeta^4$ ($z, v \in \mathbf{C}$ and $w \in \mathbf{R}$) and the deformation $p(\zeta) = a + b\zeta - \bar{a}\zeta^2, a \in \mathbf{C}, b \in \mathbf{R}$. An $SL(2,\mathbf{C})$ transformation: $\zeta \to \frac{a\zeta+b}{-\bar{b}\zeta+\bar{a}}$, $|a|^2 + |b|^2 = 1$ can be used to restrict $p(\zeta) = \tilde{b}\zeta$. For generic values of the five real parameters in $\eta(\zeta)$, one gets eight points on an elliptic curve, $\gamma^2 = \eta(z\zeta)$, corresponding to the roots of $\eta(\zeta) + p^2(\zeta) = 0$ - the divisor for four of these eight should correspond to $\Gamma(L^m)$, where L^m are holomorphic line bundles over $T\mathbf{CP}^1$ with $e^{\frac{-m\xi}{\zeta}}$ (ξ being a fiber coordinate) the transition functions. The splitting of the eight roots into two groups of four each is determined by the following condition: $\left(\int_\alpha^\infty + \int_\beta^\infty - \int_{-\frac{1}{\bar{\alpha}}}^\infty - \int_{-\frac{1}{\bar{\beta}}}^\infty\right) \frac{d\zeta}{\sqrt{\eta(\zeta)}} = 2$; the roots of $\eta(\zeta) + p^2(\zeta) = 0$ are $\alpha, \beta, -\frac{1}{\bar{\alpha}}, -\frac{1}{\bar{\beta}}$. Now, if x_1, x_2 are roots of $\eta(\zeta) + \tilde{b}\zeta = 0$ *after* the $SL(2,\mathbf{C})$ transformation:

$$\begin{pmatrix} \eta(\zeta) \\ p(\zeta) \end{pmatrix} \xrightarrow{\zeta \to \frac{a\zeta+b}{-\bar{b}\zeta+a}} \begin{pmatrix} \frac{r_1\zeta^3 - r_2\zeta^2 - r_1\zeta}{(-\bar{b}\zeta+\bar{a})^4} \\ \frac{\tilde{b}(a\zeta+b)}{(-\bar{b}\zeta+\bar{a})} \end{pmatrix},$$

$r_1, r_2 \in \mathbf{R}$, then the aforementioned constraint can be rewritten in terms of inverse elliptic functions:

$$\mathcal{P}^{-1}\left(x_1 - \frac{r_2}{3r_1}\right) + \mathcal{P}^{-1}\left(x_2 - \frac{r_2}{3r_1}\right) - \mathcal{P}^{-1}\left(-\frac{1}{\bar{x}_1} - \frac{r_2}{3r_1}\right) - \mathcal{P}^{-1}\left(-\frac{1}{\bar{x}_2} - \frac{r_2}{3r_1}\right) = \frac{m\sqrt{r_1}}{2},$$

where the inverse elliptic function $\mathcal{P}^{-1}(z) \equiv \int_\infty^z \frac{d\zeta}{\sqrt{4\zeta^3 - g_2\zeta - g_3}}$. The constraint is also equivalent to $\frac{\partial F}{\partial w} = 0$ for a suitable constraint function F defined in terms of appropriate contour integrals. The Kähler potential is then given in terms of the Legendre transform of $F : K(z, \bar{z}; \frac{\partial F}{\partial v}, \frac{\partial \bar{F}}{\partial \bar{v}}) = F(z, \bar{z}; v, \bar{v}; w) - v\frac{\partial F}{\partial v} - \bar{v}\frac{\partial \bar{F}}{\partial \bar{v}}$ evaluated at the constraint: $\frac{\partial F}{\partial w} = 0$.

Thus, one sees the existence of (inverse) elliptic functions in the hyper-Kähler metrics in twistor spaces obtained using deformations of Atiyah-Hitchin spaces and Legendre transforms.

To summarize, we have obtained the relevant flow equations for uplifting half-flat manifolds to Spin(7) eight-folds by two methods - first, by considering a double G_2 (constructed from the half-flat) -fibration over an interval, and the second, by considering a fibration of the half-flat over the product

of two intervals. We were able to explicitly uplift the Iwasawa at the standard complex structure limit in the moduli space of almost complex structures on the Iwasawa. We gave a plausibility argument against the same for the second method, at the "edge". We also gave motivating reasons for considering singular uplifts involving doubly periodic functions - the physical interpretation of the same is not yet clear.

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